MOS Revisited

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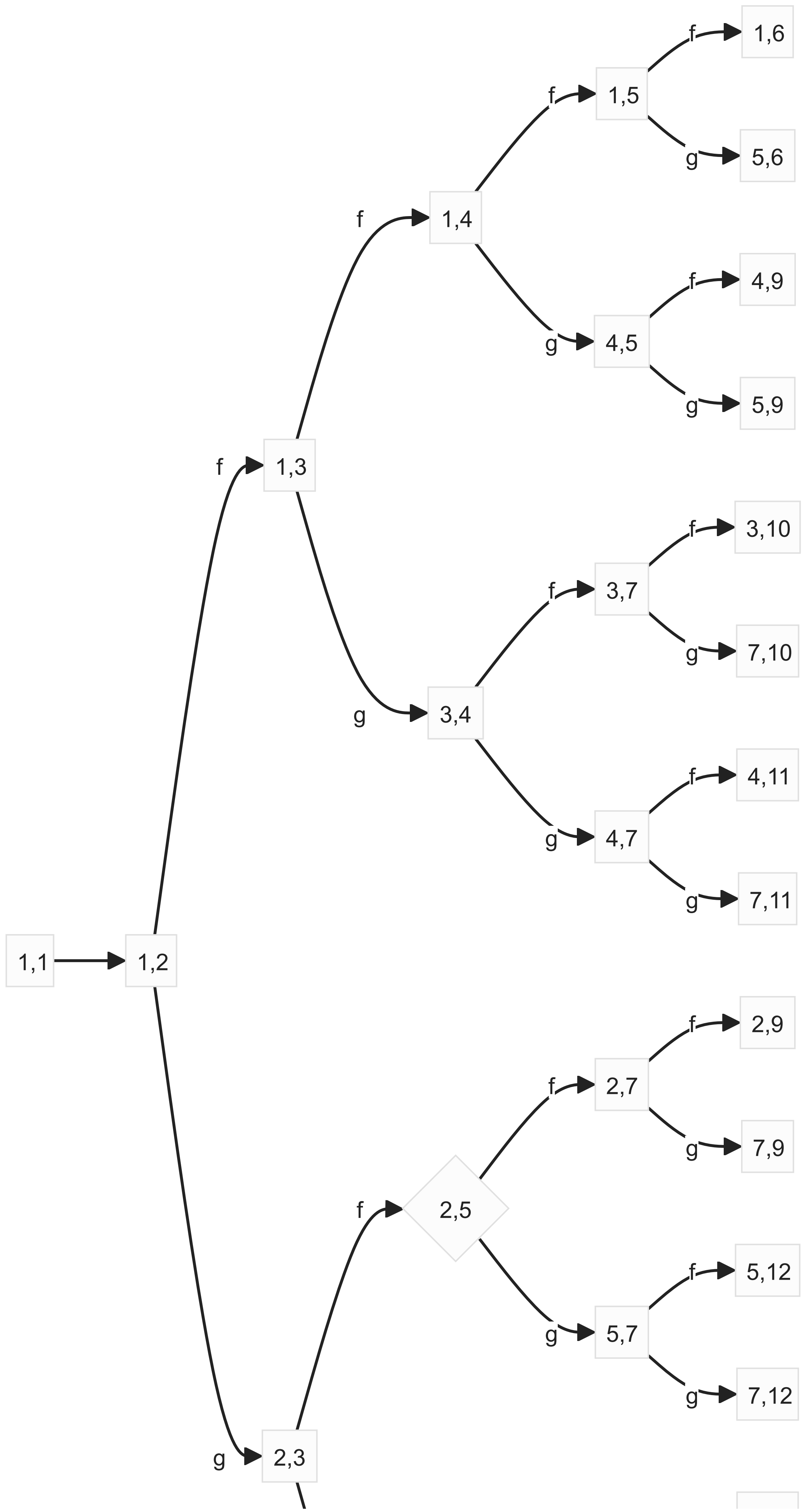
Erv Wilson derived and discussed a taxonomy of scales resultant from the Moments-of-symmentry (MOS) construction. As is the case with any taxonomy, it is limiting in that it hides the true unifying principles and logic of the universe of all binary scales. Here I outline an alternative way of constructing the MOS that is simpler and more general than the original formulation.

By simply defining the two parameters a and b (denoting the number of note steps of given sizes A and B) without reference to whether one interval is greater or smaller, we gain generality and can see the hidden logic behind scales. Not only do we reduce the number of cases to consider by a factor of 2. If we lift the requirement that intervals need to be positive, we can even introduce a transformation procedure mapping between different MOS scales. This will allow us to map the full universe of binary scales (i.e. scales consisting of two step sizes) onto the diatonic universe of Western music that most of us are familiar with. This in turn allows musicians to comfortably explore a variety of MOS scales while leveraging their experience playing on the piano keyboard, and can potentially serve as a valuable tool for composition in general.

To show how such a mapping can be achieved, we first draw the coprime-tree, a graph of pairs (a, b) in which a and b are coprime, i.e. they do not share any prime factors. We draw arrows between nodes that map $f : (a, b) \to (a, a + b)$ and $g : (a, b) \to (b, a + b)$ starting from the node $(1, 1)$ (while removing the duplicate $(1, 2)$ after the first step). Continued to infinity this procedure generates all possible pairs of coprime numbers.



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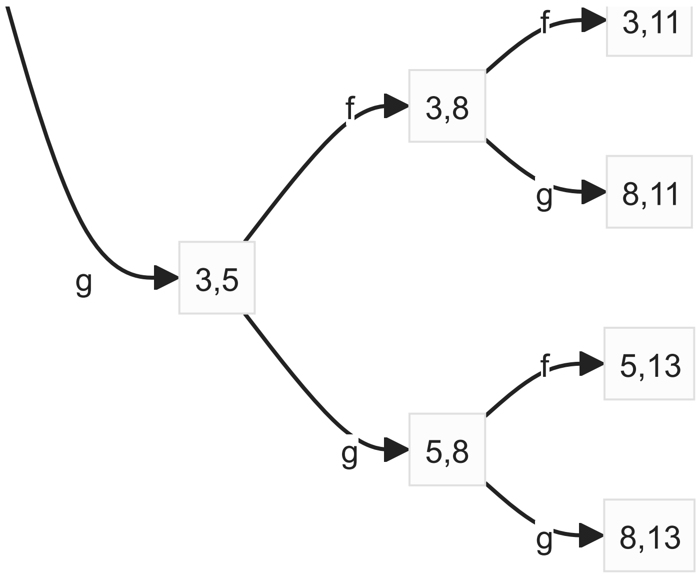
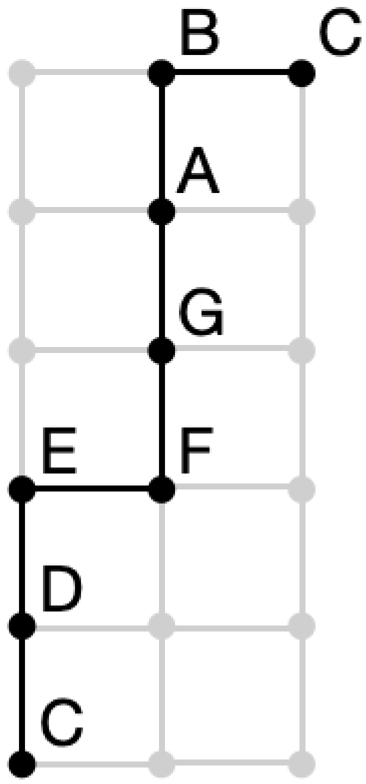


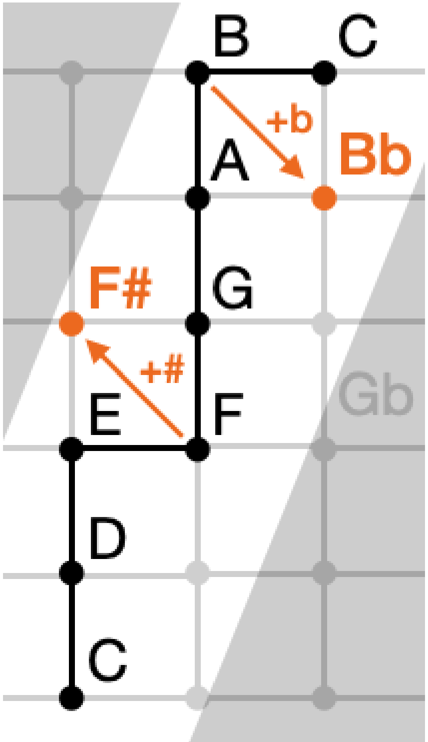
A resulting rule is that the step size with the smaller number of ocurences is always noted first. The Western diatonic scale thus belongs to the node $(2, 5)$, denoting 2 and 5 notes of each size, respectively, resulting in diatonic scales with 7 notes total.

In the following we show how we can easily transform between MOS scales. Each transition can be viewed as a linear transformation on the lattice of the two coordinates, and we can move back and forth on this tree graph, mapping scales between each other.

Let us begin by drawing a Western diatonic scale (C-Major) onto a lattice of the two coordinates a and b (using the x and y-axes respectively).

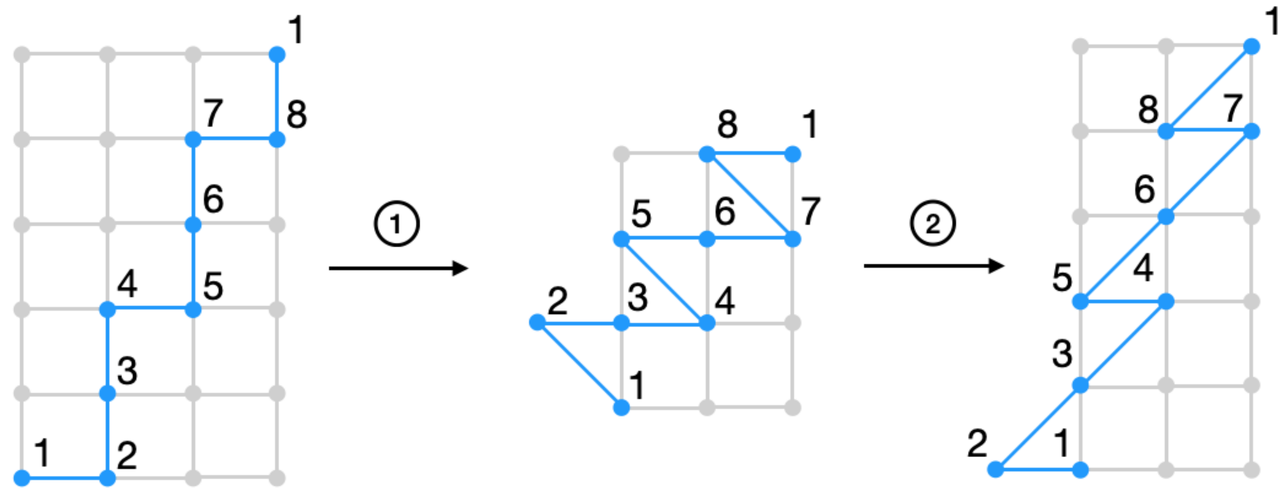


This lattice extends into infinity on every side, and we assign notes to each node. The grey node above the E, for instance, is a whole step above the E, thus is an F#. Adding accidentals thus is moving diagonally: Adding a sharp accidental to a note is equivalent to moving up and left, i.e. adding $(-1, 1)$ to a note, whereas adding a flat is moving down and right, i.e. adding $(1, -1)$. On this lattice, there is no identification of F# with Gb, they are two different notes, and in general (i.e. temperaments other than 12-TET) have different pitches. Furthermore, we can add an arbitrary number of accidentals to any note, thus having note-labels for every node on our lattice. Also note that there is a direction of enharmonic equivalence (which is in the direction $(-2, 1)$) and a direction of constant pitch, which is $(-2, 1)$ in the case of the 12-TET, but is arbitrary in general. Any value for the direction of constant pitch given by $(-\alpha, 1)$ with $\alpha > 1$ yields a tuning in which the size of interval A is smaller than that of B. Pythagorean tuning, all Meantone temperaments and all TET's can be readily described by particular choices of α. E.g. 31-TET is recovered by $\alpha = 5/3$.

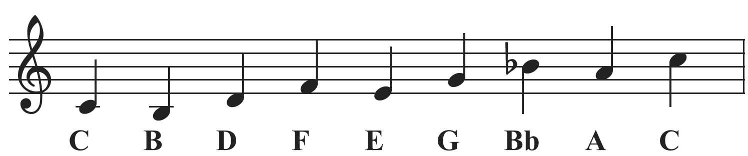


Also note the highlighted and shaded areas. Assigning notes on a piano keyboard is tantamount to a projection of notes falling inside the highlighted strip, that covers parts of the lattice, onto the one-dimensional piano keyboard. The strip runs parallel to the line connecting a note with its octave and has a width that accommodates exactly twelve notes per octave. When mapping notes from the Western diatonic system to the piano keyboard, there is a choice to be made: Will the key between the F and the G be assigned an F# or a Gb? The highlighted piano strip makes this decision visually explicit. By moving the strip to the left or to the right one can make a choice that fits the needs of the pieces performed.

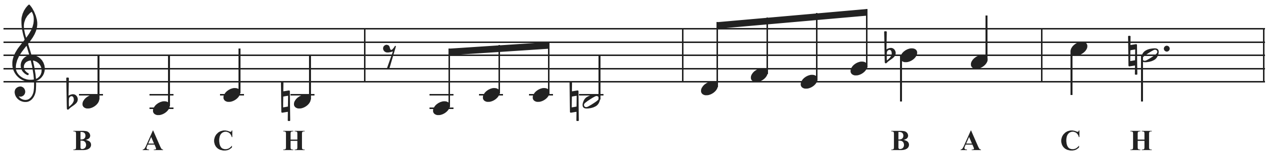
Since this $(a, b) = (2, 5)$ system is what we are used to, it is desirable to understand the other scales in the context of this particular system. Since walking the coprime tree depicted above is just linear transformations, we can transform any of the scales in our generalized MOS into any other, and in particular into the one we are used to. We demonstrate how this can be achieved using an example. We arbitrarily pick a specific scale from the $(a, b) = (3, 5)$ system, with 8 notes per octave, 3 steps of size A and 5 steps of size B, distributed as ABBABBAB.



Step (1) is to go from coprime-tree-node $(3,5)$ to $(2,3)$. It is the inverse of the transform $g:(a,b)\to(b,a+b)$ that we used to go from $(2,3)$ to $(3,5)$ in the coprime-tree. That inverse is the linear transform $g^{-1}:(a,b)\to(b-a,a)$. If we apply it to each note on our $(3,5)$-lattice and draw the path correspondingly, we thus have transformed our scale into the $(2,3)$-lattice. (Note that every note-node in one lattice gets mapped to one in any other. This must be so: The sums that appear in f and g can only yield integers, and both maps are bijective.) This yields the middle figure, which shows the ABBABBAB pattern in the $(2,3)$ system. Step (2) is to move from node $(2,3)$ to $(2,5)$ in the coprime-tree, by using the transform $f:(a,b)\to(a,a+b)$. This yields the figure to the right. Consider our transformed scale. We can observe that we can go from C to C in 8 steps, with 3 steps going down a minor second and 5 steps going up a minor third. ($5\cdot3-3\cdot1=12$ semitones and $5\cdot2-3\cdot1=7$ diatonic steps.) The transformed scale yields the following sequence of (classical $(2,5)$-system) notes: C-B-D-F-E-G-Bb-A-C. It is a 8-note $(3,5)$-diatonic scale transformed into our usual Western $(2,5)$-diatonic world.

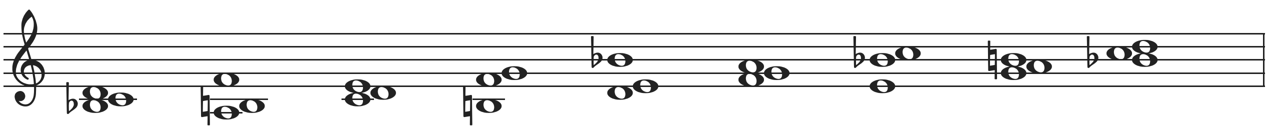


By using a different mode of the same scale, starting on the B, we can readily recover the famous B-A-C-H motif, and see that it in fact belongs to walking the notes of a two-note scale from the $(3,5)$ system in sequential (logical, not pitch) order (subtitled in German notation):



Take note that simply by walking this scale we obtain a pleasing melodic line. Its appeal is based exactly on the property that it only uses two different intervals. Our mapping between scales thus can well turn out to be a practical tool of composition. We can speculate that J.S. Bach himself might have been aware of these kinds of structures. We could walk this scale in a similar way as usual melodic lines are walking a Western diatonic scale, up and down the scale.

Because of the structural similarity between all the MOS scales, there are thirds of exactly two different sizes, defined by the interval sequences AB and BB, which in $(2,5)$ world map to a major second and a diminished fifth. The stack of third triads thus are



which does not sound convincing in standard Western tuning. (Maybe we could use stacks of fourths? They again come in two sizes only, namely ABA and ABB, amounting to an augmented first and a perfect fourth. This does not sound convincing, either.)

Nevertheless, all seven diatonic notes of the C-Major scale are part of our new B-A-C-H scale, as well, thus we can build all the usual triads (plus some using the B-flat), they are just not built from every second note of the scale. Besides, if one would decide to re-tune (which is a simple rotation of the direction of equal pitch in our lattice) in such a way that all steps in our $(3,5)$ system are into the positive pitch direction (i.e. choosing the direction of constant pitch with $\alpha < -1$), it is plausible that one could find a tuning in which stacks of thirds have a pleasing sound. It also could make sense to keep the mapping to the piano while doing the re-tuning. This would result in an assignment of pitches to the keys on the piano keyboard that is not increasing from left to right anymore. Nevertheless it would still have the important structural property of being consistent.

A different way of re-tuning the piano keyboard could prove more fruitful. One could re-assign the pitches of the Western system to different keys in such a way that playing the white keys plus a chosen black one in sequential order would yield our $(2,5)$-mapped $(3,5)$-diatonic scale. This way, playing melodies consisting of ascending or descending sequences along the new scale would be very easy. In any case of retuning, the resulting non-sequential order of pitches along the piano keyboard could need some getting-used-to, but it would be straightforward to familiarize oneself to a particular mapped scale.

The real benefit, however, will come from applying both transforms at once. That way, one can map sequential tunings of arbitrary MOS scales to the piano keyboard (given they have less than 12 notes per scale) and one has $12-(a+b)$ keys left for key changes.

Changing keys works analogously in any of the MOS scales: Raising or lowering a single note will shift the cyclic pattern. For instance, in our $(3,5)$ scale we can raise the 2nd note, which transforms the pattern ABBABBAB (which we could call $(3,5)$-dorian) into BABABBAB. Raising an accidental in the $(3,5)$-system mapped into Western diatonic world amounts to going from C a minor third up, thus Eb would become the second note of our scale instead of the B. (The $(2,5)$-interval of raising a $(3,5)$-note is thus a diminished fourth.) The $(3,5)$-dorian of the changed key would start at 4. Similarly, lowering the 7th note of the scale (replacing Bb with F#, a diminished fourth down) yields the pattern ABBABABB, which would make the 6th the tonic of the new key. We have similar modulation spaces in any of the MOS-scales, and by mapping them onto the piano keyboard we thusly have made them accessible to piano players and composers.

Considering our re-tuned keyboard, having 8 pitches in our scale means we can, given 12 notes per octave, change key 4 times (compared to 5 in the usual Western diatonic case of 7 diatonic notes per octave and non-degenerate, i.e. non-12-TET tuning.)

The combinatorial richness of tuning an instrument in one MOS scale and transforming an arbitrary other one into it offers quite some space for exploration and idea generation. Together with the suggested mapping onto the piano keyboard makes it accessible to many musicians.

It is also noteworthy that the introduced mapping procedure implies a mapping of conventional musical notation onto arbitrary scales. If one can give up the idea that notes higher on the vertical axis should have higher pitch (which is not strictly the case, anyway as E# > Fb), our conventional musical notation thus accommodates arbitrary binary scales. Together with a tuning prescription (specifying the direction of constant pitch and the size of the octave), the Western notation system thus, at least in theory, is sufficient for the whole MOS universe (and more). Whether such an extension of the usage of our notation conventions is practically feasible stays to be proven by time.

As a side note, the analogy between (a, b)-lattices and isomorphic keyboards is evident. Consistent note layouts can simply be transformed between each other by arbitrary affine transforms of the lattice. I have implemented the PitchGrid , which in the hopefully not too distant future will have all the scales discussed here as a virtual online isomorphic keyboard and also will work as a MIDI-effect that translates a standard MIDI signal (e.g. from a MIDI controller with a piano keyboard) into an MPE signal (to control an MPE enabled VST synth, for example) via the WebMIDI standard, thus enabling experimentation with arbitrary binary scales.

Besides practical concerns, a major objection to exploring the MOS universe is the concern that non-$(2, 5)$ scales might not sound as well as the Western diatonic system when used with instruments with (almost) harmonic spectra. While this certainly is true, (the Western diatonic system is by construction a good fit to harmonic spectra, as it originates from the Pythagorean tuning procedure,) new technology allows us to utilize synthesizers that make it possible to micro-tune the partials of a sound. This effectively allows us to reverse the logic *spectrum* $\rightarrow$ *diatonic system* and enables the construction of sounds specifically tuned to an arbitrary MOS scale in a particular tuning of choice. Remember that, in general, the spectrum of a resonating physical body is not harmonic. There is considerable research in this direction, notably by William A. Searles.

There is no perfect harmony. Scales, tunings and timbre are matters of expression of emotion, and can be significant part of the story the musician tells. Understanding the foundations of tonal structure and sound allows to make conscious choices and will reflect in the quality of compositions. I hope I can contribute toward making different scales more accessible for the practitioner.

Peter